CALCULATING THE NONSTATIONARY FLOW AROUND A LATTICE OF ARBITRARY PROFILES WHICH VIBRATE WITH AN ARBITRARY PHASE SHIFT

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The nonstationary motion of a single wing of arbitrary form with constant circulation has been studied in detail by Chaplygin [1] and Sedov [2].

The present paper considers the unsteady motion of an ideal incompressible liquid around a lattice of profiles of arbitrary form. Perturbation of the liquid flow is achieved through vibration and, in the general case, by small amplitude distortion of the blades. The blades in the lattice vibrate synchronously, but with an arbitrary phase shift. The case of constant phase shift from blade to blade is treated in detail.

Vibrations of profiles with a constant circulation are considered exactly; for those with a variable circulation a quasistationary technique is used.

Under exact conditions we solve also the problem of steady flow around the lattice with varying circulations around the profiles.

Examples are given of the calculation of flow due to the vibration of circles in the lattice and of profiles of a gas turbine.

1. Statement of the problem. Let us consider a lattice, consisting of arbitrary profiles (Fig. 1), in the plane of a complex variable $z = x + iy$. Let the lattice axis coincide with the y axis; the spacing of the profiles will be designated by t.

We will study the unsteady, potential motion of an ideal incompressible liquid in an infinitely connected area G, which is the space outside the above lattice.

Let us assume that an infinite, regular flow from the left strikes the lattice, while the lattice profiles undergo arbitrary vibrations with a small amplitude. First, we will consider a case in which all the profiles vary synchronously with frequency ω, but with an arbitrary, constant phase shift α between adjacent profiles.

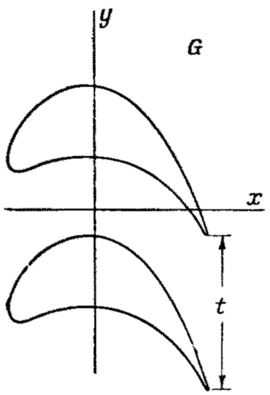

Fig. 1.

Let us introduce the complex velocity potential

$$f = \varphi + i\psi, \quad u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{1.1}$$

Here u and v are the components of the velocity w of the liquid; the velocity potential $\phi = \phi(x, y, \tau)$ and the stream function $\psi = \psi(x, y, \tau)$ depend on the coordinates and on time τ. We divide the functions ϕ and ψ into sums of the two functions

$$\varphi = \varphi_0(x, y) + \varphi_1(x, y, \tau), \qquad \psi = \psi_0(x, y) + \psi_1(x, y, \tau) \tag{1.2}$$

ϕ_0 and ψ_0 do not depend on time, and they solve the problem of the steady flow past the lattice.

This problem may be considered solved. In the rest of this paper it is assumed that it is solved by considering a cascade of circles.

The functions $\phi_1(x, y, \tau)$ and $\psi_1(x, y, \tau)$ are the velocity potential and the stream function of the full perturbed motion of the liquid caused by the vibration of the profiles. The function ϕ_1 is conveniently represented in a form analogous to the Kirchhoff form [3]

$$\varphi_1 = U\varphi_{01} + V\varphi_{02} + \Omega\varphi_{03} + \Gamma\varphi_{04} + \varphi_{05} \tag{1.3}$$

U and V are the velocity components of vibrations of an arbitrary profile along the axes of a fixed system of coordinates, Ω is the angular rotation velocity of the profile and Γ is the circulation of the velocity around the profile. These values depend only on time. Functions ϕ_{01}, ϕ_{02}, ϕ_{03} and ϕ_{04} depend only on the coordinates of the point at which the velocity potential is calculated.

The function ϕ_{05} is the potential of the flow around the profiles brought about by vortical traces which, according to the Thomson theorem, appear when there is a change in the circulation. It is obvious that ϕ_{05} depends both on the coordinates and on time. Below we consider flow with constant circulation using an exact procedure or flow with a circulation which varies with time; always, however, under quasistationary conditions ($\phi_{05} = 0$), which is permissible when low Strouhal numbers are involved.

Correction for the influence of ϕ_{05} may, in view of the linearity of the problem, may be introduced by superposition.

The boundary conditions at the profiles around which the liquid flows state the equality at appropriate points of the normal components of velocity of the contour and the liquid. These conditions are conveniently written in terms of the function ψ. The value of the function ψ at a vibrating profile, found from the requirement $\partial\psi/\partial n = w_n$, must satisfy the condition [2]

$$\psi = Uy - Vx - \frac{1}{2}\Omega(x^2 + y^2) + \text{const} \tag{1.4}$$

At an infinite distance from the lattice, disturbances caused by vibrations of the profiles should vanish. Only when the profiles vibrate in phase and with a change in circulation will the disturbances at infinity remain finite on the right side.

The solution of the problem consists in determining the complex potential which satisfies condition (1.4) at the vibrating profiles. The velocity field is found by means of (1.1); the pressure field is represented by the Lagrange integral

$$p = p_0(\tau) - \rho\frac{\partial\varphi}{\partial\tau} - \frac{1}{2}\rho(u^2 + v^2) \tag{1.5}$$

The forces and moments acting on a profile are conveniently calculated according to the general formulas of Sedov for unsteady motion.

2. Derivation of the basic formulas. Before tackling the solution, let us derive certain general formulas for a special function having those characteristics which the complex velocity potential in our problem must have.

In area G we introduce the function $F(z, t, a)$, of a complex variable z, dependent on two real parameters t and a, and having the following characteristics:

1. The function $F(z, t, a)$ satisfies the condition of generalized periodicity in the following sense:

$$F(z + imt) = e^{-jma}F(z) \qquad (m = \pm 1, \pm 2, \pm 3 \ldots) \tag{2.1}$$

Here j is an imaginary unit not interacting with imaginary unit i.

2. The function $F(z, t, \alpha)$ has no singular points in area G.

3. The function

$$F(z, t, \alpha) \to 0 \quad \text{for } x \to \pm\infty \qquad (\alpha \neq 0) \tag{2.2}$$

Let us give an integral representation of the function $F(z, t, \alpha)$.

Applying Cauchy's formula in the infinitely connected area G, taking into account conditions (2.2), we obtain

$$F(z) = \frac{1}{2\pi i} \sum_{m=-\infty}^{+\infty} \oint_{L_m} \frac{F_m(\zeta)\, d\zeta}{z - \zeta} \tag{2.3}$$

Here $F_m(\zeta)$ denotes the boundary value of the function $F(z)$ at mth contour L_m, comprising a lattice; the integration in (2.3) is carried out for all contours of the lattice. Let us restrict integration to refer only to that integration along contour L_0, which will hereafter be designated as basic. Substituting into (2.3) the variables $\zeta = \zeta_0 + imt$ and taking advantage of the fact that, according to condition (2.1),

$$F_m(\zeta) = e^{-jm\alpha} F_0(\zeta_0)$$

we obtain (the index 0 in the integration variable is disregarded)

$$F(z) = \frac{1}{2\pi i} \sum_{m=-\infty}^{+\infty} \oint_{L_0} \frac{F_0(\zeta)\, e^{-jm\alpha} d\zeta}{z - \zeta - imt} \tag{2.4}$$

Interchanging in (2.4) the order of integration and summation (which is permissible in this case), we obtain

$$F(z) = \frac{1}{2\pi i} \oint_{L_0} \left[\frac{1}{z-\zeta} + \sum_{m=1}^{\infty} \left(\frac{e^{jm\alpha}}{z - \zeta + imt} + \frac{e^{-jm\alpha}}{z - \zeta - imt} \right) \right] F_0(\zeta)\, d\zeta \tag{2.5}$$

The expression in brackets may be transformed and summed

$$\frac{1}{z-\zeta} + \frac{2(z-\zeta)}{t^2} \sum_{m=1}^{\infty} \frac{\cos m\alpha}{m^2 + (z-\zeta)^2/t^2} - \frac{2ij}{t} \sum_{m=1}^{\infty} \frac{m \sin m\alpha}{m^2 + (z-\zeta)^2/t^2} = \frac{\pi}{t} \frac{\operatorname{ch}(\pi - \alpha) z / t}{\operatorname{sh} \pi z / t} - ij \frac{\pi}{t} \frac{\operatorname{sh}(\pi - \alpha) z / t}{\operatorname{sh} \pi z / t} \tag{2.6}$$

We then obtain the final integral formula which expresses the values of the function $F(z)$ in area G through its values on the basic contour

$$F(z) = \frac{1}{4i} \oint_{L_0} F_0(\zeta)\, \Phi(z - \zeta, \alpha, q)\, d\zeta \tag{2.7}$$

We have introduced $q = 2/t$ and the function $\Phi(z, \alpha, q)$

$$\Phi(z, \alpha, q) = q\left(\frac{\operatorname{ch} {}^1/_2 (\pi - \alpha) qz}{\operatorname{sh} {}^1/_2 \pi qz} - ij \frac{\operatorname{sh} {}^1/_2 (\pi - \alpha) qz}{\operatorname{sh} {}^1/_2 \pi qz}\right) \tag{2.8}$$

The function $\Phi(z, \alpha, q)$ has the following basic characteristic:

$$\Phi\left(z + \frac{2im}{q}\right) = e^{-jm\alpha}\Phi(z) \tag{2.9}$$

The function $\Phi(z, \alpha, q)$ has simple poles at the points $2im/q$ and breaks down into the simple fractions

$$\Phi(z, \alpha, q) = \frac{1}{z} + q\sum_{m=1}^{\infty}\left(\frac{e^{jm\alpha}}{qz + 2im} + \frac{e^{-jm\alpha}}{qz - 2im}\right) \tag{2.10}$$

The function

$$\Phi(z, \alpha, q) \to \pm(1 \mp ij)\, e^{\mp \alpha q x} \quad \text{for } x \to \pm\infty \tag{2.11}$$

Let us note the special cases

$$\begin{aligned}
&\Phi(z, 0, q) = q \operatorname{cth} {}^1/_2 \pi qz + \text{const} && \text{where } \alpha = 0 \\
&\Phi(z, \pi, q) = q \operatorname{csch} {}^1/_2 \pi qz && \text{where } \alpha = \pi \\
&\Phi(z) = {}^1/_2 \pi/z && \text{where } q = 0\ (t = \infty)
\end{aligned}$$

Let us expand $\Phi(z - \zeta, \alpha, q)$ in (2.7) into a series of powers of ζ

$$\Phi(z - \zeta, \alpha, q) = \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^n}{n!}\, \frac{d^n\Phi(z)}{dz^n} \tag{2.12}$$

Interchanging the order of summation and integration, we then obtain from (2.7), (2.8) and (2.12)

$$F(z) = \frac{1}{4i}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} N_{n+1} \frac{d^n\Phi(z)}{dz^n} \qquad \left(N_n = \frac{1}{2\pi i}\int F(\zeta)\, \zeta^{n-1} d\zeta\right) \tag{2.13}$$

We obtain an expansion $\Phi(z, \alpha, q)$ in the vicinity of the poles. First of all, we note that, on the basis of property (2.9), the expansion in the vicinity of a pole m differs from the expansion in the vicinity of $m = 0$ by only the multiple $\exp(-jm\alpha)$.

First we expand into a Taylor series (in the vicinity of the pole $m = 0$) the expression in parentheses in (2.10)

$$\cos m\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} q^{2k}}{2^{2k-1} m^{2k}} z^{2k-1} - ij \sin m\alpha \sum_{k=1}^{\infty} \frac{(-1)^k q^{2k+1}}{2^{2k} m^{2k+1}} z^{2k+1} \tag{2.14}$$

We disregard here the constant which is unessential for our purposes.

Substituting this series in (2.10) and changing the order of summation, we find the expansion of $\Phi(z, \alpha, q)$ in the vicinity of the pole $m = 0$

$$\Phi(z, \alpha, q) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{c_k q^{2k}}{2^{2k-1}} z^{2k-1} - ij \sum_{k=1}^{\infty} \frac{s_k q^{2k+1}}{2^{2k}} z^{2k} \tag{2.15}$$

Under these conditions

$$c_k = c_k(\alpha) = (-1)^{k+1} \sum_{m=1}^{\infty} \frac{\cos m\alpha}{m^{2k}}, \qquad s_k = s_k(\alpha) = (-1)^k \sum_{m=1}^{\infty} \frac{\sin m\alpha}{m^{2k+1}} \tag{2.16}$$

The series (2.15) must converge all the way to the nearest singular point; i.e. whenever $|z| < t = 2/q$.

To obtain an expansion of the function $F(z, a, q)$ into a Laurent series, we differentiate (2.15) n times with respect to z. After some transformations, we obtain

$$\frac{d^n\Phi(z)}{dz^n} = \frac{(-1)^n n!}{z^{n+1}} + \sum_k^{\infty} \frac{q^{2k} c_k}{2^{2k-1}} \frac{(2k-1)!}{(2k-n-1)!} z^{2k-n-1} - $$
$$- ij \sum_k^{\infty} \frac{s_k q^{2k+1}}{2^{2k}} \frac{(2k)!}{(2k-n)!} z^{2k-n} \tag{2.17}$$

The summation over k is carried out here in such a way that the powers of z are greater than zero (unessential constants may be disregarded).

On the basis of (2.13) and (2.17), $F(z)$ in the vicinity of the pole $m = 0$ may now be represented by the series (intermediary transformations omitted)

$$F(z, \alpha, q) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)! N_n}{z^n} + \sum_{n=0}^{\infty} \sum_k^{\infty} \frac{(2k-1)!\, q^{2k} c_k}{n!\, 2^{2k-1}} N_{2k-n} z^n - $$
$$- ij \sum_{n=0}^{\infty} \sum_k^{\infty} \frac{(2k)!\, q^{2k+1} s_k}{n!\, 2^{2k}} N_{2k-n+1} z^n \tag{2.18}$$

The series must converge when $|z| < t = 2/q$.

3. The unsteady flow around a lattice of circles oscillating with a phase shift. Consider a lattice of circles with radii $r = 1$ oscillating with small amplitude.

The law governing the oscillations of any circle in the lattice may be written in the following manner:

$$U = U_0 e^{j(\omega\tau - m\alpha)}, \qquad V = V_0 e^{j(\omega\tau - m\alpha)} \tag{3.1}$$

The stream function at the circumference must then, according to (1.4), assume the following form

$$\psi = U_0 \sin\theta - V_0 \cos\theta$$

Here θ is the polar angle; the multiplier dependent on time is omitted.

Since the exterior of the lattice of circles is chosen as a canonical region, let us examine immediately the more general case with a boundary value ψ

$$\psi = \sum_{n=1}^{\infty} \delta_n \cos n\theta + \sum_{n=1}^{\infty} \gamma_n \sin n\theta \tag{3.2}$$

The complex velocity potential f of an unsteady flow without circulation around a lattice of circles should have the same characteristics as the function $F(z, a, q)$. Consequently, outside a lattice of circles, the complex potential should be expanded into a functional series (2.13) and, in the vicinity of the pole $m = 0$, into a Laurent series (2.18).

The expansion of $F(z, a, q)$ into a Laurent series in the vicinity of pole m differs from expansion (2.18), as we should expect from (2.1), by only the multiple $\exp(-jm\alpha)$.

Consequently, the problem consists in the determination of the coefficients N_n in (2.18), needed to satisfy boundary condition (3.2).

Ordinarily, coefficients N_n must be complex numbers, each with two imaginary units

$$N_n = (A_n + jB_n) + i\,(C_n + jD_n) \tag{3.3}$$

Substituting (3.3) in (2.18) and separating the imaginary part with respect to i, we obtain a series into which the stream function ψ may be expanded in the vicinity of pole $m = 0$.

Comparing the boundary values with those given in (3.2), we arrive at four systems of infinite equations, from which four series of unknowns A_n, B_n, C_n and D_n may be found:

$$(-1)^{n-1}(n-1)!\,C_n + \sum_k^{\infty} \frac{(2k-1)!\,q^{2k} c_k}{n!\,2^{2k-1}}\, C_{2k-n} + \sum_k^{\infty} \frac{(2k)!\,q^{2k+1} s_k}{n!\,2^{2k}}\, B_{2k-n+1} = \delta_n$$

$$-(-1)^{n-1}(n-1)!\,A_n + \sum_k^{\infty} \frac{(2k-1)!\,q^{2k} c_k}{n!\,2^{2k-1}}\, A_{2k-n} - \sum_k^{\infty} \frac{(2k)!\,q^{2k+1} s_k}{n!\,2^{2k}}\, D_{2k-n+1} = \gamma_n$$

$$(-1)^{n-1}(n-1)!\,D_n+\sum_k^{\infty}\frac{(2k-1)!\,q^{2k}c_k}{n!\,2^{2k-1}}\,D_{2k-n}-\sum_k^{\infty}\frac{(2k)!\,q^{2k+1}s_k}{n!\,2^{2k}}\,A_{2k-n+1}=0 \tag{3.4}$$

$$(-1)^{n-1}(n-1)!\,B_n+\sum_k^{\infty}\frac{(2k-1)!\,q^{2k}c_k}{n!\,2^{2k-1}}\,B_{2k-n}+\sum_k^{\infty}\frac{(2k)!\,q^{2k+1}s_k}{n!\,2^{2k}}\,C_{2k-n+1}=0$$

Ignoring the question of the convergence of the process of successive approximations in general, let us note that if $B_n = D_n = 0$, the systems belong to a fully regular class.

We observe that the systems fall into two groups of two unknown series apiece, with unknowns with an even index connected with unknowns of another series with an odd index.

In practice successive approximations converge very quickly. Calculations are substantially reduced by the diagonal symmetry of the coefficients.

The distribution of the velocities at the circumferences in the lattice is found through the function

$$v_s = -\,\partial\psi/\partial r \qquad \text{for } r = 1 \tag{3.5}$$

Here v_s is the circumferential component of the absolute velocity of the liquid.

Using (3.5), (2.18) and (3.4) we obtain, after transformations which we omit here,

$$\begin{aligned} v_s = \cos(\omega\tau - m\alpha)\sum_{n=1}^{\infty}\{[2(-1)^{n-1}n!\,C_n - n\delta_n]\cos n\theta\, - \\ -[2(-1)^{n-1}n!\,A_n + n\gamma_n]\sin n\theta\}\, - \\ -2\sin(\omega\tau - m\alpha)\sum_{n=1}^{\infty}(-1)^{n-1}n!\,[D_n\cos n\theta - B_n\sin n\theta] \end{aligned} \tag{3.6}$$

Formula (3.6) gives the solution of the problem, since it describes the distribution of the velocities at any circumference in the lattice (m is the number of the circumference) and at any moment in time.

The radial velocities v_r on a circumference are known quantities according to the specifications of the problem.

Since at a circumference $v_s = (\partial\phi/r\,\partial\theta)_{r=1}$, we obtain from (3.6)

$$\varphi = \cos(\omega\tau - m\alpha) \sum_{n=1}^{\infty} \{[2(-1)^{n-1}(n-1)!\, C_n - \delta_n] \sin n\theta + \\ + [2(-1)^{n-1}(n-1)!\, A_n + \gamma_n] \cos n\theta\} - \\ - 2\sin(\omega\tau - m\alpha) \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!\, [D_n \sin n\theta + B_n \cos n\theta] \quad (3.7)$$

The distribution of pressures is determined from the Lagrange equation (1.5) from the known distribution of the potentials (3.7) and the square of the full velocity, equal to $w^2 = u^2 + v^2 = v_s{}^2 + v_r{}^2$.

The field of velocities outside of the lattice may be determined from (2.7) or by (2.13).

Example. Let us find the distribution of velocity for the perturbation of a liquid flow caused by the vibration of circles in the lattice in the direction of its axis with a velocity $V = V_0 \exp j\omega\tau$, $V_0 = 1$.

Given the parameter $q = 2/\tau = 0.7$. We carry out calculations for phase shifts $a = 0, .., \pi/4$. From (1.4) and (3.2) it follows that $\delta_n = \gamma_n = 0$, except that $\delta_1 = -1$.

a) For $a = 0$ we find, according to (2.16)

$$c_1 = 1.645, \quad c_2 = -1.082, \quad c_3 = 1.017, \quad c_4 = -1.004 \ldots, \quad s_n = 0$$

It is obvious from (3.4) that $A_n = B_n = D_n = 0$ and that $C_{2n} = 0$, which is in conformity with the considerations of flow symmetry. The coefficients C_{2n-1} are determined from the solution of the first system (3.4)

$$C_1 = -0.714, \quad C_3 = -0.0117 \ldots$$

The distribution of the absolute velocity v_s at the circumferences is determined from (3.6)

$$v_s = -(0.428 \cos\theta + 0.129 \cos 3\theta + \ldots) \cos\omega\tau$$

b) For $a = \pi$ we find, according to (2.16),

$$c_1 = -0.824, \quad c_2 = 0.946, \quad c_3 = -0.985, \quad c_4 = 0.998, \ldots, \quad s_n = 0$$

It is obvious from (3.4) that $A_n = B_n = D_n = 0$ and that $C_{2n} = 0$. The coefficients C_{2n-1} are determined from the solution of the first system (3.4)

$$C_1 = -1.25, \quad C_3 = 0.0192 \ldots$$

The absolute velocity v_s at the circumferences is determined from (3.6)

$$v_s = (-1.52 \cos \theta + 0.230 \cos 3\theta - \ldots) \cos (\omega\tau - m\pi)$$

c) For $\alpha = \pi/4$, according to (2.16)

$$c_1 = 0.575, \quad c_2 = -0.694, \quad c_3 = 0.706, \quad c_4 = -0.707\ldots$$
$$s_1 = -0.845, \quad s_2 = 0.740, \quad s_3 = -0.715, \quad s_4 = 0.709\ldots$$

From (3.4) it follows that

$$A_n = D_n = 0, \qquad B_{2n-1} = C_{2n} = 0$$

The coefficients C_{2n-1} are determined from the solution of the first and fourth systems of equations (3.4)

$$C_1 = -0.895, \quad C_3 = -0.00795\ldots, \quad B_2 = -0.0675, \quad B_4 = 0.00337\ldots$$

The distribution of the absolute velocity v_s at the circumferences is determined according to (3.6)

$$v_s = -(0.790 \cos \theta + 0.095 \cos 3\theta + \ldots) \cos (\omega\tau - {}^1\!/_4)m\pi + $$
$$+ (0.270 \sin 2\theta - 0.162 \sin 4\theta + \ldots) \sin (\omega\tau - {}^1\!/_4\, m\pi)$$

It is obvious that with such a flow there will be eight groups of circumferences, at which the distribution of velocities at any given moment in time will be different. Figure 2 gives curves for the four characteristic values $\beta = \omega\tau - 1/4\, m\pi$.

4. Purely circulatory flow around a lattice of circles with a phase shift. Let us examine the case of purely circulatory, steady flow around an immovable lattice of circles under the condition that the circulation around the mth circle is equal to $\Gamma_m = \Gamma_0 e^{jm\alpha}$

It is clear from the analysis in Sections 1 and 2 that the complex velocity potential for a given flow may be represented in the following manner:

$$f = \frac{\Gamma_0}{2\pi i} \int \Phi(z, \alpha, q)\, dz + \sum_n \frac{(-1)^n}{n!} N_{n+1} \frac{d^n\Phi(z, \alpha, q)}{dz^n} \tag{4.1}$$

The expansion of the first member (4.1) in the vicinity of the pole $m = 0$ is achieved by integrating (2.15)

$$\int \Phi(z)\, dz = \ln z + \sum_{n=1}^{\infty} \frac{c_n q^{2n}}{2^{2n} n} z^{2n} - ij \sum_{n=1}^{\infty} \frac{s_n q^{2n+1}}{2^{2n}(2n+1)} z^{2n+1} \tag{4.2}$$

In the flow we are presently considering, the circumferences of the lattice represent the stream lines and, consequently, the stream function at these circumferences assumes constant values.

The coefficients N_n in (4.1) will be double complex numbers of the form (3.3). In view of the considerations of flow symmetry $A_n = D_n = 0$ and $B_{2n} = D_{2n-1} = 0$. We will substitute (2.18) and (4.2) into (4.1) and, separating the part imaginary with respect to i, we equate it to zero. We obtain two systems of infinite equations from which we determine two series of unknowns C_{2n} and B_{2n-1}

$$-(n-1)!\,C_n + \sum_k^{\infty} \frac{(2k-1)!\,c_k q^{2k}}{2^{2k-1}\,n!} C_{2k-n} + \sum_k^{\infty} \frac{(2k)!\,s_k q^{2k+1}}{2^{2k}\,n!} B_{2k-n+1} = \frac{q^n c_{n/2}}{2^{n-1} n} \qquad (4.3)$$

$$-(n-1)!\,B_n + \sum_k^{\infty} \frac{(2k-1)!\,c_k q^{2k}}{2^{2k-1}\,n!} B_{2k-n} + \sum_k^{\infty} \frac{(2k)!\,s_k q^{2k+1}}{2^{2k}\,n!} C_{2k-n+1} = \frac{q^n c_{(n-1)/2}}{2^{2n-1}\,n}$$

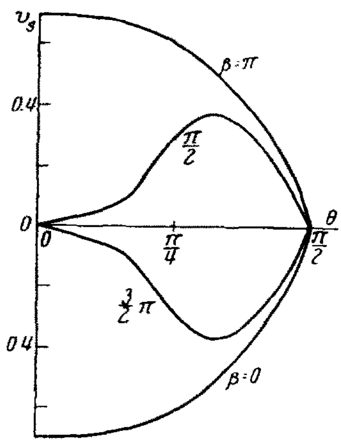

Fig. 2.

On the basis of (4.1), (2.18), (4.2), (3.3) and (4.3) we obtain a series into which the complex potential of purely circulatory flow in the vicinity of pole $m = 0$ can be expanded

$$f = \frac{\Gamma_0}{2\pi i}\left[\ln z + i \sum_{n=2,4,6\ldots}^{\infty} (n-1)!\,C_n (z^n - z^{-n}) + j \sum_{n=1,3,5\ldots}^{\infty} (n-1)!\,B_n (z^n + z^{-n})\right] \qquad (4.4)$$

Multiplying this expression by $\exp(-jm\alpha)$, we obtain an expansion in the vicinity of pole m and, separating the real part (relative with respect to both imaginary units), we obtain the distribution of the velocity potential at the circumferences ($r = 1$)

$$\varphi = \frac{\Gamma_0}{2\pi}\left[\theta - 2\cos m\alpha \sum_{n=2,4,6\ldots}^{\infty} (n-1)!\,C_n \sin n\theta - 2\sin m\alpha \sum_{n=1,3,5\ldots}^{\infty} (n-1)!\,B_n \cos n\theta\right] \qquad (4.5)$$

The distribution of the velocities is obtained by differentiating with respect to θ

$$v_s = \frac{\Gamma_0}{2\pi}\left[\cos m\alpha - 2\cos m\alpha \sum_{n=2,4,6\ldots}^{\infty} n!\,C_n \cos n\theta + 2\sin m\alpha \sum_{n=1,3,5\ldots}^{\infty} n!\,B_n \sin n\theta\right] \qquad (4.6)$$

When examining a quasistationary problem (and in a canonical region) we must replace sin ma and cos ma by sin $(\omega\tau - ma)$ and cos $(\omega\tau - ma)$.

We note that, as follows from (2.13), the velocity at an infinite distance from the lattice generally tends to zero. Only when $a = 0$ does the velocity at infinity $x \to \pm\infty$ have values which are equal, though opposite in sign.

Example. Let us determine the distribution of velocities at the circumferences in a lattice of circles when circulation changes from circumference to circumference with a phase shift equal to: $a = 0$, π and $\pi/4$. The parameter $q = 2/\tau = 0.7$, $\Gamma_0/2\pi = 1$.

a) For $a = 0$ we obtain $s_n = 0$ and $B_n = 0$. From (4.3) we obtain

$$C_2 = 0.184, \qquad C_4 = 0.00012\ldots$$

Then, from (4.6)

$$v_s = 1 + 0.736\cos 2\theta - 0.0057\cos 4\theta + \ldots$$

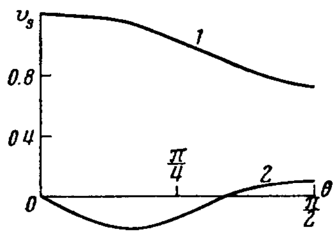

Fig. 3.

b) For $a = \pi$ we obtain $s_n = 0$ and $B_n = 0$. From (4.3) we obtain

$$C_2 = 0.110, \qquad C_4 = 0.000945\ldots$$

Then, from (4.6)

$$v_s = (1 - 0.441\cos 2\theta + 0.046\cos 4\theta - \ldots)\cos m\pi$$

c) For $a = \pi/4$ we obtain

$$C_2 = -0.0060, \qquad C_4 = 0.00073\ldots$$
$$B_1 = 0.00753, \quad B_3 = 0.0175\ldots$$

(the coefficient B_3 must be greater than the remaining coefficients B_n)

$$v_s = (1 + 0.240\cos 2\theta - 0.035\cos 4\theta + \ldots)\cos {}^1/_4 m\pi + \\ + (-0.015\sin\theta - 0.210\sin 3\theta - \ldots)\sin {}^1/_4 m\pi$$

Figure 3 gives curves for the distribution of velocity along circumferences $m = 0$ and $m = 2$. The circumference $m = 2$ has a flow with a circulation equal to zero. There are a total of eight groups of circumferences with different velocity distributions.

5. Flow around a lattice of arbitrary profiles vibrating with a phase shift. Let us consider the problem of the vibration of profiles in a lattice with a phase shift. We assume the form of the profiles to be arbitrary. If the profiles oscillate with a change in circulation, the problem is solved by a quasistationary procedure.

The velocity potential $\phi(x, y, \tau)$ and the stream function $\psi(x, y, \tau)$ satisfy the Laplace equation; for this reason we use the conformal transformation method.

As a canonical region we select the area outside the lattice of circles in the plane z.

The relationship between the lattice areas is established by a functional series [4]

$$\zeta = az + \frac{\pi}{t}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a_{-(n+1)}\frac{d^n}{dz^n}\operatorname{cth}\frac{\pi z}{t} \tag{5.1}$$

This series is a special case of the functional series (2.13).

In the vicinity of the poles (5.1) is expanded into the series

$$\zeta = az + \sum_{n=1}^{\infty}\frac{a_{-n}}{z^n} + \sum_{n=0}^{\infty}a_n z^n \tag{5.2}$$

The coefficients of the regular part of the series depend on those of the principal part

$$a_n = \sum_{k}^{\infty}\frac{(-1)^n 2^{2k}B_k}{2k(2k-n-1)!\,n!}\left(\frac{\pi}{t}\right)^{2k}a_{-(2k-n)} \tag{5.3}$$

Having set $a = a' + ia''$; $a_n = a_n' + ia_n''$; $a_{-n} = a_{-n}' + ia_{-n}''$, we may give the contour of the profile in the lattice by two parametric equations

$$\begin{aligned} \xi &= a'\cos\theta + \sum_{n=1}^{\infty}(a_{-n}' + a_n')\cos n\theta + \sum_{n=1}^{\infty}(a_{-n}'' - a_n'')\sin n\theta \\ \eta &= a''\sin\theta + \sum_{n=1}^{\infty}(a_{-n}'' + a_n'')\cos n\theta + \sum_{n=1}^{\infty}(a_n' - a_{-n}')\sin n\theta \end{aligned} \tag{5.4}$$

θ is the polar angle of the basic circle in the lattice.

The method for determining the coefficients a_n and a_{-n} is described in [4] and we will therefore consider the series (5.4) as given.

Let us assume that the blades in the lattice achieve arbitrary (small amplitude) bending and torsional vibrations with an arbitrary constant phase shift α.

Now we determine (1.4) the boundary values of the stream function $\psi(\xi, \eta)$ at the profiles in the lattice. Substitution there of the parametric equations (5.4) gives us the boundary values $\psi(\theta)$ at the circumferences in the canonical region in the form of a trigonometric series (3.2). The corresponding problem in the canonical region is then solved.

The distribution of velocities from the noncirculatory (3.6) and the purely circulatory (4.6) flows at a basic circumference is known. The magnitude of the circulation around the basic circumference is established from the Chaplygin-Zhukovskii postulate for a quasistationary flow.

The above may be extended without difficulty to the lattice, whose profiles not only vibrate but are also subject to small deformations. The boundary value for the function ψ of the deformation motion of normal velocity v_n is established by the relationship $\partial\psi/\partial s = v_n(\theta, r)$.

The above problems have involved a phase shift changing from blade to blade by a constant amount. All of the results can be generalized, which leads merely to an increase in the calculating labor.

Instead of the basic function $\Phi(z, a, q)$, we may examine a series of functions $\Phi(z, a_r, q)$, in each of which a_r = const. Then, in the expansions (2.17) and (4.2) a sign of summation over r is added in front, while the numbers c and s will be summed by two indices k and r.

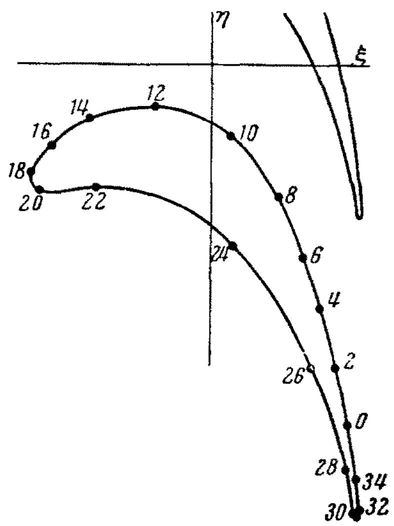

Fig. 4.

Example. Let us find the distribution of velocities at the profiles in a lattice (Fig. 4), which vibrate with a phase shift $a = \pi$ in a direction perpendicular to the chord.

We omit the preliminary calculations, which are carried out as described above.

Let the components of the vibration velocities along the axes of the coordinates be $U_0 = V_0 = 1$.

The coefficients of series (3.2), which represents the boundary values $\psi(\theta)$ at the circumference in the lattice, are

$$\delta_1 = -1.020, \quad \delta_2 = +0.161, \quad \delta_3 = +0.0717$$

$$\delta_4 = -0.0574, \quad \delta_5 = +0.0119$$

$$\gamma_1 = +0.505, \quad \gamma_2 = +0.129, \quad \gamma_3 = -0.0542$$

$$\gamma_4 = +0.0188, \quad \gamma_5 = +0.0093$$

Vibrations occur with a phase shift $a = \pi$; therefore, it follows from (2.16) that $s_k = 0$, and we then find, from (3.4), that $B_n = D_n = 0$.

Substituting δ_n and γ_n in (3.4) solving the resulting equations (q = 0.853), we obtain C_n and A_n (we keep terms with an index no higher than 5).

$$C_1 = -1.535, \quad C_2 = -0.209, \quad C_3 = +0.0950, \quad C_4 = +0.0120, \quad C_5 = +0.000358,$$
$$A_1 = -0.385, \quad A_2 = +0.110, \quad A_3 = +0.0134, \quad A_4 = +0.00411, \quad A_5 = -0.000275$$

Further, we solve the first system of Equations (4.3) and obtain

$$C_2^\circ = +0.188 \frac{\Gamma_0}{2\pi}, \qquad C_4^\circ = -0.00479 \frac{\Gamma_0}{2\pi} \ldots$$

where the upper index denotes that these coefficients refer to purely circulatory flow.

Using the coefficients thus calculated we obtain the distribution of velocities along the circumference by means of (3.6) and (4.6).

Applying the Chaplygin-Zhukovskii condition, we obtain a circulation $\Gamma_0/2\pi = 2.41$. The final expression of the law governing the distribution of absolute velocities v_s along the circumference is then given by the series

$$v_s(\theta) = \sum_n C_n^* \cos n\theta + \sum_n A_n^* \sin n\theta + 2.41$$

In this series

$$C_1^* = -2.050, \quad C_2^* = -1.300, \quad C_3^* = +0.925, \quad C_4^* = +0.210$$
$$A_1^* = +0.265, \quad A_2^* = 0.172, \quad A_3^* = 0, \quad A_4^* = +0.160$$

Figure 5 shows the distribution of absolute velocities at profile v_s with respect to the maximum velocity of the vibration of the profile

$$V_k = \sqrt{U_0^2 + V_0^2}$$

An evolute curve of the profile is drawn along the abscissa. The numbers of the points on the abscissa correspond to the numbers of the points on the profile (Fig. 4). A significant peak in the velocities appears in the case of flow near the leading edge. A separation of the flow curves occurs between points 0 and 34.

This graph is an auxiliary one.

Figure 6 gives a representation of how the vibration of the profile affects the regime of the normal flow around it. The solution is obtained by addition of the perturbation flow calculated above, caused by the vibration of the profiles, and the regular flow (we omit the calculation) around the lattice. In constructing curves for the distribution of velocity $v_s/w_{1\infty}$, it is assumed that the maximum vibration velocity is 0.05 of the velocity of the flow striking the lattice.

With such low Strouhal numbers as are typical in such practical problems, the application of the quasistationary theory produces an entirely insignificant error.

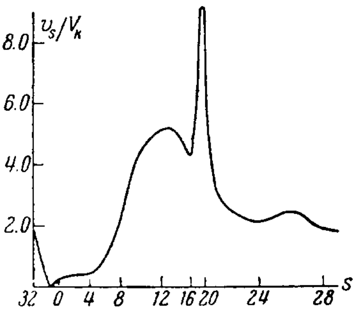

Fig. 5.

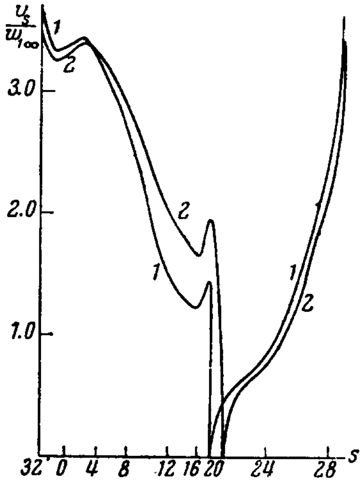

Fig. 6.

In Fig. 6, curve 1 corresponds to the moment in time when the profile, during vibration, becomes convex; curve 2 when it becomes concave. A particularly pronounced change is observed at the leading edge. The forward critical point shifts along the profile, while the rear point is specified as fixed.

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